Convection-Dominated Accretion Flows

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ABSTRACT

Non-radiating, advection-dominated, accretion flows are convectively unstable. We calculate the two-dimensional ($r-\theta$) structure of such flows assuming that (1) convection transports angular momentum inwards, *opposite* to normal viscosity and (2) viscous transport by other mechanisms (e.g., magnetic fields) is weak ($\alpha \ll 1$). Under such conditions convection dominates the dynamics of the accretion flow and leads to a steady state structure that is marginally stable to convection. We show that the marginally stable flow has a constant temperature and rotational velocity on spherical shells, a net flux of energy from small to large radii, *zero* net accretion rate, and a radial density profile of $\rho \propto r^{-1/2}$, flatter than the $\rho \propto r^{-3/2}$ profile characteristic of spherical accretion flows. This solution accurately describes the full two-dimensional structure of recent axisymmetric numerical simulations of advection-dominated accretion flows, but its relevance to "real" flows is less certain.

1. Introduction

Analytical calculations of the structure of non-radiating, advection-dominated, accretion flows (ADAFs) have shown that they are convectively unstable in the radial direction (e.g., Begelman & Meier 1982; Narayan & Yi 1994).[2] In the absence of radiation, the entropy of the gas increases as it accretes (due to viscous dissipation) and the sub-Keplerian rotation of the flow is insufficient to stabilize this unstable entropy gradient. This conclusion has been confirmed by several numerical simulations (Igumenshchev, Chen, & Abramowicz 1996; Igumenshchev & Abramowicz 1999, hereafter IA; Stone, Pringle, & Begelman 2000, hereafter SPB).

Narayan & Yi (1994; 1995) argued that convection was unlikely to significantly modify the structure of ADAFs, essentially because the inflow time of the gas would be shorter than the characteristic convective turnover time (i.e., advection would overwhelm convection). Their analysis assumed that convection behaved like "normal" viscosity, transporting angular momentum and energy from small to large radii in the flow.

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[2] Note that this is very different from thin accretion disks which can be vertically convective.

A number of treatments of convection in thin accretion disks, however, have argued that convection does not act like normal viscosity; instead it transports angular momentum inwards, opposite to what is needed to allow accretion (Ryu & Goodman 1992; Stone & Balbus 1996; Balbus, Hawley, & Stone 1996). Narayan, Igumenshchev, & Abramowicz, (2000; hereafter NIA) have shown that, if present in ADAFs, this inward angular momentum transport by convection can dramatically modify the structure of the accretion flow, essentially suppressing accretion. In this paper we reach the same conclusion as NIA via a different analysis; we also find the full two-dimensional structure of the flow (NIA considered height integrated flows).[3]

We assume that convection transports angular momentum inwards and that outward transport by other mechanisms (e.g., magnetic fields) is weak in the sense that the induced radial velocities are highly subsonic (the Shakura-Sunyaev dimensionless viscosity or its magnetic analogue is small; $\alpha \ll 1$). In this case the steady state structure of the accretion flow must, statistically speaking, approach marginal stability to convection. This is because, in the absence of marginal stability, the characteristic convective turnover time is of order the dynamical time, much shorter than the inflow time of the gas (because $\alpha \ll 1$). Convection therefore dominates the flow structure and forces it to marginal stability (just like in the solar convection zone).

In the next section (§2) we calculate the two-dimensional $(r - \theta)$ structure of an axisymmetric, radially self-similar accretion flow that is marginally stable to convection. This provides us with the angular structure of the flow. In §3 we argue that the only marginally stable accretion flow which satisfies mass, angular momentum, and energy conservation is one for which there is no accretion ($\dot{M} = 0$), but a net flux of energy from small to large radii.[4] This is in contrast to standard ADAF solutions in which there is no net energy flux, but a finite accretion rate (see, e.g., Narayan & Yi 1994; Blandford & Begelman 1999). In §4 we compare our solution to recent numerical simulations and discuss its relevance to "real" ADAFs.

2. Marginal Stability: the Angular Structure

We consider stability to local, axisymmetric, adiabatic, perturbations. Entropy gradients in ADAFs tend to destabilize perturbations while rotation stabilizes them. The net stability criterion can be found by requiring that the projection of the buoyancy and centrifugal forces on the displacement vector of a mode is negative (so there is a net restoring force for any perturbation); this

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[3]The generalization to 2D is important because the flows under consideration are not thin; the vertical scale height is of order the radius. A 2D analysis therefore generates additional confidence in the viability of the solution (see the discussion in Narayan & Yi 1995, which generalized the original ADAF treatment of Narayan & Yi 1994 to 2D).

[4]In the self-similar limit, a finite energy flux is generated by a "black hole" of infinitesimal radius which accretes an infinitesimal amount of matter.

yields (e.g., Begelman & Meier 1982)

$$(\nabla s \cdot \mathbf{dr})(\mathbf{g} \cdot \mathbf{dr}) - \frac{2\gamma v_\phi}{R^2}\left(\nabla[v_\phi R] \cdot \mathbf{dr}\right) dR \leq 0. \tag{1}$$

In equation (1), $R = r\sin\theta$ is the cylindrical radius, $\mathbf{dr} = dr\hat{r} + rd\theta\hat{\theta}$ is the displacement vector, $s = \ln(p) - \gamma\ln(\rho)$ is $(\gamma - 1)$ times the entropy, $\mathbf{g} = -\hat{r}v_K^2/r + \hat{R}v_\phi^2/R$ is the effective gravity, and $v_K = r^{-1/2}$ and v_ϕ are the Keplerian and rotational velocities, respectively (taking $GM = 1$, where M is the mass of the central object).

Away from the boundaries, there is only one characteristic length scale in an ADAF, the distance from the central object (r). We therefore consider radially self-similar flows in which the dynamical variables are power laws in radius times functions of θ, i.e., $\rho = r^{-n}\rho(\theta)$, $c^2 = r^{-1}c^2(\theta)$, $p = r^{-1-n}p(\theta)$, and $v_\phi^2 = r^{-1}v_\phi^2(\theta)$; c^2 is the isothermal sound speed (i.e., temperature) and $p = \rho c^2$ is the pressure. The radial scalings of the temperature and rotational velocity are set by the $1/r$ gravitational potential. For reasons that will become clear in §3 we allow the density to be an arbitrary power law in radius, $\rho \propto r^{-n}$, rather than requiring the usual spherical flow value of $n = 3/2$.

In the Appendix we show that, for a given n and γ, the requirement that the flow be marginally stable to convection for all θ uniquely determines its structure.[5] The temperature and rotational velocity are constant on spherical shells, i.e., independent of θ, with values

$$c^2 = \frac{v_K^2}{\frac{\gamma+1}{\gamma-1} - n} \quad \text{and} \quad v_\phi^2 = 2v_K^2\left(\frac{\frac{1}{\gamma-1} - n}{\frac{\gamma+1}{\gamma-1} - n}\right). \tag{2}$$

The density is then a power law in $\sin\theta$, with

$$\rho(\theta) \propto (\sin\theta)^{2(\frac{1}{\gamma-1} - n)}. \tag{3}$$

3. Implications of Marginal Stability: the Radial Structure

The marginally stable solution derived in the previous section describes the angular structure of the flow for a given γ and n. Here we argue that the requirement of marginal stability, together with mass, angular momentum, and energy conservation, also uniquely fixes the radial density power law, n. We first give a physical argument and then elaborate on it with some non-rigorous mathematics.

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[5]Marginal stability means that the quadratic form given by equation (1) is negative along all directions but one, along which it is equal to zero; see the discussion below equation (A5) in the Appendix.

Since radiation is assumed negligible, we write the conservation laws as

$$\frac{d\bar{F}}{dr} = 0, \quad \text{where} \quad \bar{F} \equiv \int_0^\pi d\theta \sin\theta F. \tag{4}$$

F represents the time-averaged radial flux of energy (F_E), angular momentum (F_L), and mass (F_M) at a given r and θ, while $\bar{F}$ is the corresponding angle-integrated flux (i.e., the net flux through a spherical shell of radius r).

In standard ADAF models, $n = 3/2$; as discussed below, this corresponds to a finite flux of mass onto the central object ($\bar{F}_M < 0$), but zero net angular momentum and energy flux, i.e., $\bar{F}_E = \bar{F}_L = 0$ (Narayan & Yi 1994; Blandford & Begelman 1999). That such a solution is possible can be understood as follows. Viscosity gives rise to a net torque on the gas which drives it inwards ($\bar{F}_M < 0$). The outward angular momentum flux due to this torque is exactly balanced by the inward flux of angular momentum due to the bulk motion of the gas; this is how $\bar{F}_L = 0$. Associated with the outward flux of angular momentum due to the viscous stress is an outward flux of energy due to the work done by the stress on the gas. The net energy flux is zero, i.e., $\bar{F}_E = 0$, because the outward energy flux due to viscosity is exactly balanced by the inward flux of energy due to the bulk motion of the gas. The latter is given by $F_M Be$, where Be is the Bernoulli constant,

$$Be = \frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}c^2 - v_K^2, \tag{5}$$

where $v^2 = v_\phi^2 + v_r^2 + v_\theta^2$. For standard ADAF solutions, Be must be positive and relatively large (i.e., a reasonable fraction of v_K^2) in order for the inward mechanical flux of energy to balance the outward viscous flux (Narayan & Yi 1994; Blandford & Begelman 1999).

From equation (2) one can readily show that the Bernoulli constant satisfies $Be \approx 0$ *for all* θ, γ *and* n in the marginally stable state. Thus, regardless of the magnitude or sign of the mass flux ($\bar{F}_M$), there is no energy flux due to bulk motion in a flow that is marginally stable to convection. It is therefore not possible to have a normal ADAF, i.e., an $n = 3/2$ solution. There is no inward mechanical flux of energy to balance the outward flux due to viscosity.[6]

Physically, the above argument shows that a marginally stable flow must have a net flux of energy to large radii. This uniquely fixes n for a given viscosity prescription since there is only one n for which the outward flux of energy is independent of radius, as is required in steady state. Modeling viscosity via an $r-\phi$ component of the stress tensor, the viscous energy flux is $\propto r^2 \nu \rho v_\phi^2/r$, where ν is the kinematic viscosity. For a self-similar flow, the viscous flux is $\propto \nu\rho \propto \nu r^{-n}$ and so if $\nu \propto r^\beta$, a finite energy flux requires $n = \beta$. For a Shakura-Sunyaev viscosity prescription, $\nu = \alpha c r \propto r^{1/2}$, so that $n = 1/2$. In this marginally stable solution with a finite energy flux, $\bar{F}_M = 0$, i.e., there is no accretion; matter simply circulates in convective eddies. This is possible because the inward transport of angular momentum by convection balances the outward transport by viscosity.

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[6]Convection strengthens the argument since it gives rise to an additional outward flux of energy; see below.

To elaborate on this discussion, we construct specific expressions for F_L and F_E. This elaboration is intended solely to motivate the various signs utilized in the above argument. We neither can nor do construct a fully satisfactory model for the energy and angular momentum fluxes in a turbulent/convective accretion flow (see NIA for an analysis using mixing length theory). The reader satisfied with the above argument may wish to proceed directly to §4.

We consider a model in which the energy and angular momentum fluxes have contributions from two sources: (1) hydrodynamic effects such as bulk motion of the gas and convection and (2) non-hydrodynamic viscous processes such as those due to magnetic fields. We model the latter with an $r\phi$-component of the stress tensor, which we denote $-T^v$, where $T^v > 0$ because "normal" viscosity transports angular momentum outwards. In the usual Shakura-Sunyaev prescription, $T^v = 1.5\nu\rho v_\phi/r$, where $\nu = \alpha c r$ is the kinematic viscosity.

With this model, the radial fluxes can be written as

$$F_E = \langle r^2 \rho v_r Be\rangle + r^2 \langle v_\phi\rangle T^v \tag{6}$$

$$F_L = \langle \rho v_r r^3 \sin\theta v_\phi\rangle + r^3 \sin\theta T^v, \tag{7}$$

$$F_M = \langle r^2 \rho v_r\rangle, \tag{8}$$

where $\langle\ \rangle$ denotes a time average over many convective turnover times. The first terms on the right hand side of equations (6) and (7) are the hydrodynamic contribution to the energy and angular momentum flux; these will be explained in more detail below. The second terms on the right hand side of equations (6) and (7) are those due to the viscous stress T^v. Associated with the angular momentum flux due to this stress is an energy flux $\propto \Omega T^v$ due to the rate at which the stress does work on the gas ($\Omega = v_\phi/[r\sin\theta]$ is the rotation rate of the gas).

To understand the meaning of the hydrodynamic contribution to the angular momentum flux in equation (7), we decompose v_ϕ into mean and fluctuating components via $v_\phi = \langle v_\phi\rangle + \tilde{v}_\phi$. With this decomposition,

$$\langle \rho v_r r^3 \sin\theta v_\phi\rangle = r\sin\theta\langle v_\phi\rangle F_M + r^3 \sin\theta T^c, \tag{9}$$

where $T^c = \langle \rho v_r \tilde{v}_\phi\rangle$ is the stress tensor associated with convection (the Reynold's stress); T^c is negative because convection is assumed to transport angular momentum inwards. Physically, the two terms in equation (9) correspond to the angular momentum flux due to bulk motion and convection, respectively.

An analogous decomposition of the hydrodynamic contribution to the energy flux in equation (6) is less useful because the energy flux contains a large number of quantities that are third order in the fluctuating variables; a detailed analysis of these terms is non-trivial and not particularly illuminating. Instead, we simply write

$$\langle r^2 \rho v_r Be\rangle = \langle Be\rangle F_M + r^2\langle v_\phi\rangle T^c + F_c, \tag{10}$$

which is formally just a definition of the convective energy flux F_c. Physically, the first two terms on the right hand side in equation (10) correspond to the energy flux due to bulk motion of the gas and the energy flux due to the rate at which the "convective stress," T^c, does work on the gas. The remaining term in equation (10), F_c, is the energy flux due to convection in the absence of bulk motion or angular momentum transport. This is, in fact, the energy flux usually considered in convective media (e.g., mixing length theory). In non-rotating stars, e.g., convection does not lead to a significant transport of mass or angular momentum, but it does lead to a significant flux of energy; in our notation this is captured by the term F_c in equation (10). Although a detailed expression for F_c in terms of the turbulent quantities is complicated, it is also unnecessary for our purposes. All we need is its sign; $F_c > 0$ because convection transports energy from high entropy (small radii) to low entropy (large radii).

Using the Shakura-Sunyaev prescription for T^v, one finds that the viscous contribution to the energy flux is $\propto r^{1/2-n}$. Similarly the viscous contribution to the angular momentum flux is $\propto r^{2-n}$. In order to conserve energy, angular momentum, and mass ($d\bar{F}/dr = 0$), the corresponding angle-integrated fluxes must either be zero or independent of radius. Therefore, we can write our final conservation equations as

$$\bar{F}_E = \langle Be \rangle \bar{F}_M + \bar{F}_c + r^2 \langle v_\phi \rangle \bar{T} = C_1 \delta_{n,1/2}, \tag{11}$$

$$\bar{F}_L = r \langle v_\phi \rangle \bar{F}_M + r^3 \overline{T \sin\theta} = C_2 \delta_{n,2}, \tag{12}$$

$$\bar{F}_M = \overline{\langle r^2 \rho v_r \rangle} = C_3 \delta_{n,3/2}, \tag{13}$$

where the C_i are constants and $T = T^v + T^c$ is the total stress tensor. In writing equations (11)-(13) we have integrated the fluxes on spherical shells and have used the fact that c^2, v_ϕ^2, and Be are independent of θ in the marginally stable state and thus can be taken out of the angular integration.

Standard ADAF solutions correspond to $n = 3/2$. As indicated by equations (11)-(13), this is a solution for which there is a finite flux of mass onto the central object, but no net energy or angular momentum flux (Narayan & Yi 1994; Blandford & Begelman 1999). As argued above, no such solution is possible when convection dominates the structure of the flow and drives it to marginal stability.

Consider a putative $n = 3/2$ solution to equations (11)-(13). For such a solution $\bar{F}_M < 0$ ($v_r < 0$) since there is an inward flux of mass. From equation (11), and using $Be \approx 0$ near marginal stability, we see that $\bar{F}_E \approx r^2 v_\phi \bar{T} + \bar{F}_c$. Both terms in this expression for $\bar{F}_E$ are positive and so it is not possible to have $\bar{F}_E = 0$.[7] There is consequently no solution with $n = 3/2$. This argument shows that an ADAF that is marginally stable to convection must have a net flux of energy to large radii. Mathematically, the corresponding self-similar solution has $n = 1/2$. In this

[7] An inward flux of mass with $\bar{F}_L = 0$ requires $\overline{T \sin\theta} > 0$. In principle, one can have $\bar{T} < 0$ with $\overline{T \sin\theta} > 0$ since we do not know the θ distribution of the convective ($T^c < 0$) and viscous ($T^v > 0$) stresses (this would correspond to an outward angular momentum flux and an inward energy flux due to "viscosity + convection"). It is likely, however, that $\overline{T \sin\theta} > 0$ because $T > 0$ at all θ, i.e., because viscosity wins out over convection ($|T^v| > |T^c|$) at all θ. In this case $\bar{T} > 0$ follows.

solution, $\bar{F}_M = 0$, i.e., there is no accretion; this is possible because the inward transport of angular momentum by convection exactly balances the outward transport by viscosity ($\overline{T \sin\theta} = 0$).

4. Discussion

Our argument can be summarized as follows. If convection in ADAFs transports angular momentum inwards and if outward transport due to other mechanisms is weak ($\alpha \ll 1$), the time averaged structure of an ADAF must approach marginal stability to convection. By analyzing the structure of a marginally stable flow we find that there must be a net outward flux of energy due to convection. For a self-similar flow there can either be a net flux of mass or a net flux of energy, but not both (§3). Thus our marginally stable flow with a finite energy flux has *zero* accretion rate in the self-similar regime; the inward angular momentum transport due to convection cancels the outward transport due to viscosity. Physically, matter in a given spherical shell circulates indefinitely in convective eddies instead of accreting (in reality there will be a small net accretion rate due to the violation of self-similarity near the surface of the central object). This marginally stable solution has a radial density profile of $\rho \propto r^{-1/2}$, rather than the usual $\rho \propto r^{-3/2}$ for spherical flows.

SPB and IA have carried out axisymmetric hydrodynamical simulations of ADAFs; they find that the flow is convectively unstable for small α. Our analytical calculation reproduces the time averaged properties of these simulations rather well. SPB's run K corresponds to a Shakura-Sunyaev like viscosity with $\alpha = 10^{-3}$; in this simulation, they find $\rho \propto r^{-1/2}$, as we derived in §3. SPB also consider non self-similar viscosity prescriptions for which $\nu = \rho$ or $\nu =$ constant. As discussed in §3, for $\nu \propto r^\beta$ a finite energy flux requires $n = \beta$ (where $\rho \propto r^{-n}$). This follows from the requirement that the viscous energy flux $\propto r^2 \nu \rho v_\phi^2 / r \propto \nu\rho$ be independent of radius. Thus for both $\nu = \rho$ ($\beta = -n$) and $\nu =$ constant ($\beta = 0$), a net energy flux requires $n = 0$, i.e., ρ independent of radius. This is precisely what is found in the simulations (see SPB's Fig. 7).

The interpretation of the numerical results as due to a marginally stable flow is supported by the excellent quantitative agreement between our results and the simulations.[8] For example, SPB find rotational velocities that are nearly constant on spherical shells (see their Figs. 7 & 12), as we derived in §2 and the Appendix.[9] For $\gamma = 1.5$, our marginally stable $n = 1/2$ solution has $c^2 = 0.22 v_K^2$, in good agreement with IA's low α simulations, which find $c^2 \approx 0.2 v_K^2$ (see their Fig. 10). In addition, our predicted angular density profiles of $\rho \propto \sin^2\theta$ (for $\gamma = 5/3$ and $n = 1/2$) and $\rho \propto \sin^3\theta$ (for $\gamma = 5/3$ and $n = 0$) are in excellent agreement with the simulations (see SPB's Fig.

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[8]SPB noted that the angular structure of the contours of entropy, density, angular momentum, and other dynamical variables found in the simulations correspond to that expected near marginal stability (see the discussion in Begelman & Meier 1982).

[9]Near the pole ($\theta \lesssim 10^o$), v_ϕ decreases in the simulations. This is because near the pole $v_\phi =$ constant corresponds to a divergent rotation rate. In reality, and in the simulations, this is smoothed out by a $\theta - \phi$ component of the viscous stress tensor not considered in this work.

12 and Fig. 7, respectively), as is our prediction that the time averaged Bernoulli constant should be small, i.e., $\ll v_K^2$ (see SPB's Figs. 4 & 9 and the discussion before their eq. [9]).

It is important to note that our analysis does not rely on the suggestion of Blandford & Begelman (1999) that ADAFs will drive strong outflows. In fact, our density profile of $\rho \propto r^{-1/2}$ is derived assuming no outflows.

A shortcoming of our analysis is that we cannot quantify how small α must be for the marginally stable solution to be valid; this is remedied in NIA. They solve height-integrated versions of equations (11)-(13) by using explicit expressions for the various fluxes involved (assuming Shakura-Sunyaev like diffusion coefficients). They show that for $\alpha \lesssim \alpha_{crit} \sim 0.1$, the $n = 1/2$ solution is the only self-similar solution. For larger α, they recover the usual $n = 3/2$ ADAF solution (consistent with IA's finding that for $\alpha \gtrsim 0.1$ the flow becomes convectively stable).

If relevant to real flows, the $\rho \propto r^{-1/2}$ density profile corresponding to the marginally stable state would have important observational implications. In particular, the X-ray to radio flux ratio and the X-ray spectrum of accreting supermassive black holes are sensitive to the variation of density with radius, i.e., n (Di Matteo et al. 2000; Quataert & Narayan 1999). Smaller n, e.g., $n = 1/2$ instead of $n = 3/2$, results in a larger X-ray to radio flux ratio and a much harder X-ray spectrum (dominated by bremsstrahlung instead of Comptonization). This appears to be consistent with several low-luminosity AGN in nearby elliptical galaxies (Di Matteo et al. 2000).

There are, however, several concerns about the relevance of the convection-dominated solution that can only be addressed by future 3D numerical simulations: (1) Is the statistical steady state we have calculated unstable to non-axisymmetric perturbations? This is important because our conclusion that there must be a net energy flux follows from the properties of the marginally stable state. (2) Does convection transport angular momentum inwards in 3D flows? If not, one is likely to get a normal ADAF, perhaps modified by strong outflows (Blandford & Begelman 1999; see also IA's high α simulations). Stone & Balbus (1996) have argued that outward angular momentum transport by convection requires coherent azimuthal pressure gradients. These are manifestly absent in the axisymmetric simulations of SPB and IA. Thus 2D convection is artificial in an important way. (3) How large is α due to magnetic fields? If $\alpha \gtrsim 0.1$, as is quite plausible, then convection will be a minor perturbation (except perhaps near the pole), as originally argued by Narayan & Yi (1994; 1995) and shown in more detail by NIA.

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A. Calculation of the Marginally Stable Solution

In this Appendix we calculate the angular structure of a marginally stable flow by analyzing equation (1) assuming radial self-similarity. Equation (1) can be written as the inequality

$$Ax^2 + By^2 + Cxy \leq 0, \tag{A1}$$

where $x = dr$ and $y = rd\theta$.

The coefficients of the quadratic form are given by

$$A = g_r \partial_r s - \frac{2\gamma v_\phi}{R^2} \sin\theta \partial_r (R v_\phi), \tag{A2}$$

$$B = \frac{g_\theta}{r} \partial_\theta s - \frac{2\gamma v_\phi}{r R^2} \cos\theta \partial_\theta (R v_\phi), \tag{A3}$$

and

$$C = g_\theta \partial_r s + \frac{g_r}{r} \partial_\theta s - \frac{2\gamma v_\phi}{R^2} \left(\cos\theta \partial_r (R v_\phi) + \frac{\sin\theta}{r} \partial_\theta (R v_\phi) \right). \tag{A4}$$

The marginally stable flow corresponds to

$$A \leq 0, \qquad B \leq 0, \quad \text{and} \quad C^2 = 4AB. \tag{A5}$$

The first two conditions follow because the flow must be stable to $x = 0$ or $y = 0$ perturbations (purely tangential and purely radial, respectively). The condition $C^2 = 4AB$ follows because the left hand side of equation (A1) defines a quadratic function with negative curvature. If it has two real roots then it is positive somewhere and hence there are perturbations which are unstable. Marginal stability occurs when the two roots collapse to one ($C^2 = 4AB$) and stability happens when the two roots of the quadratic function are imaginary ($C^2 < 4AB$).

In addition to marginal stability, the flow must satisfy radial and θ momentum balance

$$\frac{v_K^2}{r} = -\rho^{-1} \partial_r p + \frac{v_\phi^2}{r} \tag{A6}$$

and

$$\cot\theta \rho v_\phi^2 = \partial_\theta p, \tag{A7}$$

where $p = \rho c^2$ is the pressure and c is the isothermal sound speed.

With radial self-similarity, equations (A6) and (A7) become

$$1 = (1+n)c^2 + v_\phi^2 \tag{A8}$$

and

$$\cot\theta \rho v_\phi^2 = c^2 \rho' + \rho (c^2)', \tag{A9}$$

where $' \equiv \partial_\theta$ and all dynamical variables (e.g., ρ, c^2) are now dimensionless functions of θ.

For a self-similar flow the coefficients of the quadratic form become

$$A = 1 - n(\gamma - 1) + v_\phi^2 \left(n(\gamma - 1) - (\gamma + 1)\right), \tag{A10}$$

$$B = \frac{v_\phi^2 \cot\theta}{1 - v_\phi^2}\left((1-\gamma)\cot\theta v_\phi^2(1+n) - 2\gamma v_\phi v_\phi'\right) - \frac{2\gamma v_\phi}{\sin^2\theta}\left(v_\phi \cos^2\theta + v_\phi' \cos\theta \sin\theta\right), \tag{A11}$$

and

$$C = 2v_\phi^2 \cot\theta \left((\gamma - 1)n - (\gamma + 1)\right). \tag{A12}$$

The marginal stability requirement is that $C^2 = 4AB$. Defining a new independent variable, $\zeta = -\ln\sin\theta$, this can be written as an equation describing the marginally stable temperature profile

$$\frac{dc^2}{d\zeta} = \frac{(c^2 - c_1^2)(c_2^2 - c^2)}{c_3^2 - c^2}, \tag{A13}$$

where

$$c_1^2 = \frac{1}{\frac{\gamma+1}{\gamma-1} - n}, \qquad c_2^2 = \frac{1}{1+n}, \qquad c_3^2 = \frac{c_1^2 + c_2^2}{2}. \tag{A14}$$

Since the region extending from the equator to the pole corresponds to ζ going from 0 to ∞, the only non-singular global solution is $c^2 = c_1^2$. Any other solution becomes singular somewhere between the equator and the pole.

Thus the only *global* marginally stable flow is one for which c^2 and v_ϕ^2 are constant on spherical shells, i.e., independent of θ, with values

$$c^2 = \frac{1}{\frac{\gamma+1}{\gamma-1} - n}, \qquad v_\phi^2 = 2\frac{\frac{1}{\gamma-1} - n}{\frac{\gamma+1}{\gamma-1} - n}. \tag{A15}$$

From equation (A7) the density is then a power law in $\sin\theta$, with

$$\rho \propto (\sin\theta)^{2(\frac{1}{\gamma-1} - n)}. \tag{A16}$$

Equation (A15) gives the marginally stable state corresponding to $C^2 = 4AB$. One can show that this state also satisfies $A < 0$ and $B < 0$ and so all of the conditions in equation (A5) are satisfied.

There are several points to make about this solution. First, we have not applied any boundary conditions at the equator in deriving this solution, despite the fact that marginal stability is a differential equation in θ. The reason no boundary conditions are necessary is that, for a given n and γ, there is only one non-singular solution which extends from the equator to the pole.

Second, in evaluating the stability of height integrated ADAFs to convection, Narayan & Yi (1994) and NIA used $N^2 + \kappa^2 > 0$ as their stability requirement, where N is the Brunt-Vaisala frequency and κ is the epicyclic frequency. In our notation, this is equivalent to taking $A < 0$ since, at the equator, $A = -\gamma(N^2 + \kappa^2)$. In the full 2D flow, the criterion $A < 0$ is, however, a necessary, but not a sufficient, criterion for convective stability (see the discussion below eq. [A1]). Physically, $A < 0$ is the stability requirement for radial perturbations, i.e., for $y \propto d\theta = 0$ (see eq. [A1]). Near marginal stability, however, the "almost" unstable modes have $y = -\sqrt{A/B}x$ and are thus not radial perturbations. True marginal stability therefore corresponds to $C^2 = 4AB$ and not $A = 0$. It turns out that our marginally flow has a rotational velocity a factor of $\sqrt{2}$ larger than the flow defined by $A = 0$.[10] NIA found that their marginally stable flow (defined by $A = 0$) underpredicted the rotational velocities found in IA's simulations by a factor of $\approx \sqrt{2}$ (see the discussion in their §7); this is readily explained by the above analysis.[11]

[10]This additional rotation is needed to stabilize all perturbations.

[11]To be concrete, for $\gamma = 5/3$, our $n = 1/2$ solution has $c^2 = 0.28 v_K^2$, while NIA's solution gives $c^2 = 0.48 v_K^2$; the simulations find $c^2 \approx 0.25 v_K^2$, in excellent agreement with our result (see §7 and Fig. 4 of NIA).